DTP–MSU/00-05

hep-th/0005099

Solitons and black holes in non-Abelian Einstein-Born-Infeld theory

V.V. Dyadichev[*]and D.V. Gal'tsov[†]

Department of Theoretical Physics,
Moscow State University, 119899, Moscow, Russia

May 6, 2000

Abstract

Recently it was shown that the Born–Infeld–type modification of the quadratic Yang–Mills action gives rise to classical particle-like solutions in the flat space which have a striking similarity with the Bartnik-McKinnon solutions known in the gravity coupled Yang-Mills theory. We show that both families are continuously related within the framework of the Einstein-Born-Infeld theory through interpolating sequences of parameters. We also investigate an internal structure of the associated black holes. It is found that the Born–Infeld non–linearity leads to a drastic modification of the black hole interior typical for the usual Yang-Mills theory. In the latter case a generic solution exhibits violent metric oscillations near the singularity. In the Born-Infeld case a generic interior solution is smooth, the metric has the standard Schwarzschild type singularity, and we did not observe internal horizons. Such smoothing of the 'violent' EYM singularity may be interpreted as a result of quantum effects.

PACS numbers: 04.20.Jb, 04.50.+h, 46.70.Hg

[*]Email: rkf@mail.ru
[†]Email: galtsov@grg.phys.msu.su

Physical significance of the particle-like solutions to the Einstein-Yang-Mills (EYM) field equations found by Bartnik and McKinnon (BK) [1] (for more details see a review paper [2]) as well as their possible role in the string-inspired models remains rather obscure. These 'particles' where found to have a sphaleronic nature [3] and they could be responsible for fermion number violating effects. However, it is not clear whether these purely classical solutions can survive the embedding into some quantum framework. Another related puzzle is the singularity structure of the associated black holes [4, 6, 5]. It was shown that the metric inside the classical EYM black holes exhibits violent oscillations near the singularity, which go far beyond the classical bounds [7, 8, 9]. Presumably these oscillations must be regularized in the quantum theory, but no relevant explanation was suggested so far.

To probe the effect of quantum corrections to the Yang-Mills lagrangian within the string/M theory framework it is tempting to utilize the Born-Infeld modification of the Yang-Mills action describing the low energy dynamics of D-branes [10]. Classical solutions to both the Abelian and non-Abelian Born-Infeld theories received recently much attention. Although there are certain complications in the definition of the trace over the gauge group generators in the Born-Infeld action [11, 12, 13], it is believed that the simplified 'square root' form of the lagrangian gives correct (at least qualitatively) predictions concerning solitons. Magnetic monopoles were shown to persist in such a theory [14, 15, 16]. A new type of a particle-like solution in the non-Abelian Born-Infeld model was obtained by Gal'tsov and Kerner (GK) [17]. It was shown that this theory gives rise to flat space classical glueballs which have a striking similarity with the BK solutions.

More close relationship between these two particle-like configurations becomes clear within the Einstein-Born-Infeld (EBI) theory. As was shown recently by Wirschins, Sood and Kunz [18], the GK solutions survive when gravity is added, in which case the corresponding black holes also come into play. There is a substantial difference between the above two families which has to be clarified, however. Both the BK and GK particles form the same kind of discrete sequences resulting from the 'quantization' of the parameter entering the boundary conditions near the origin. But contrary to the BK case for which this parameter sequence is convergent to a limiting value, in the GK case the corresponding sequence is divergent. So it is necessary to check more carefully whether the both families are continuously related indeed.

Here we investigate the non-Abelian Einstein-Born-Infeld solitons and black holes in more detail. We show that within this framework one has a unique family of the regular particle-like solutions smoothly interpolating between the GK and BK solutions while the gravitational coupling constant varies from zero to a large value in units of the Born-Infeld 'critical field' parameter. We also investigate the interior structure of the non-Abelian EBIYM black holes and find that the problem of violent metric oscillations here is resolved indeed.

Recall that the static spherically symmetric EYM field equations admit three smooth branches of local solutions near the singularity exhibiting the Schwarzschild, Reissner-Nordström and the imaginary-charge Reissner-Nordström type behavior. Neither of these three, however, has a sufficient number of free parameters to be generic. Therefore when

one is moving from the event horizon into the black hole interior one can meet smooth local solutions near the singularity at best for discrete values of the black hole mass [7]. The generic EYM black hole interior looks very differently from other explicitly known cases. When the singularity is approached the metric exhibits oscillations with an infinitely growing amplitude and an infinitely decreasing period. Clearly, the classical bounds are exceeded after a few oscillation cycles. In the non-Abelian EBI theory we can also find several local solution branches near the singularity, but the situation is drastically different. The local solution which has a sufficient number of parameters to be generic has a perfectly smooth Schwarzschild type behavior. Therefore the problem of oscillations is resolved in a natural way.

As in [14, 17, 18] we assume the 'square root' form of the non–Abelian Born–Infeld action which, as we believe, describes well enough particle-like solutions being at the same time much simpler than the favored by strings symmetrized trace [11] action. Thus the EBIYM action is chosen as

$$S = \frac{1}{16\pi G} \int \left(-R + 4G\beta^2 (1 - \mathcal{R}) \right) \, d^4x, \tag{1}$$

where

$$\mathcal{R} = \sqrt{1 + \frac{1}{2\beta^2} F^a_{\mu\nu} F_a^{\mu\nu} - \frac{1}{16\beta^4} (F^a_{\mu\nu} \tilde{F}_a^{\mu\nu})^2}. \tag{2}$$

Without loss of generality the dimensionless gauge coupling constant (in the units $\hbar = c = 1$) will be set to unity, so we are left with two (dimensionfull) parameters: the BI 'critical field' β of dimension L^{-2}, and the Newton constant G of dimension L^2 (Planck's length). From these one can form a dimensionless constant

$$g = G\beta, \tag{3}$$

which is the only substantial parameter of the theory [18]. The decoupling of gravity corresponds to the limit $g \to 0$, while the $g \to \infty$ limit (after rescaling) gives the EYM theory.

We consider the case of the $SU(2)$ gauge group assuming for the YM field a usual spherically symmetric static purely magnetic ansatz

$$A = (1 - w(r)) \left(T_\theta \sin\theta d\varphi - T_\varphi d\theta \right), \tag{4}$$

where a rotated basis for the gauge group generators is used. The spacetime metric is parameterized as follows:

$$ds^2 = N\sigma^2 dt^2 - \frac{dr^2}{N} - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right). \tag{5}$$

The equations of motion after a rescaling of the radial coordinate $(\beta)^{1/2} r \to r$ (making r dimensionless) take the form of two coupled equations for N and w:

$$\left(\frac{N w'}{\mathcal{R}} \right)' = \frac{w(w^2 - 1)}{r^2 \mathcal{R}} - \frac{2g w'^3 N}{\mathcal{R}^2}, \tag{6}$$

$$(Nr)' = 1 + 2gr^2(1 - \mathcal{R}), \tag{7}$$

where now

$$\mathcal{R} = \sqrt{1 + 2\frac{Nw'^2}{r^2} + \frac{(1-w^2)^2}{r^4}}, \tag{8}$$

and primes denotes the derivatives with respect to r. The third is a decoupled equation for σ

$$(\ln \sigma)' = \frac{2gw'^2}{r\mathcal{R}} \tag{9}$$

which can be easily solved once the YM function w is found.

We are interested in the asymptotically flat configurations such that the local mass function $m(r)$, defined through

$$N = 1 - \frac{2m(r)}{r}, \tag{10}$$

has a finite limit $m \to M$ as $r \to \infty$, while $\sigma \to 1$. Like in the EYM case, it can be easily derived from the Eq. (7) that finiteness of M implies the following asymptotic behavior of the YM function: $w = \pm 1 + O(r^{-1})$. In this limit $\mathcal{R} \to 1$ so the BI non-linearity is negligible.

Let us first discuss the globally regular solutions which start at the origin with the following series expansion (its convergence in a non-zero domain may be proved by the standard methods [19]):

$$w = 1 - br^2 + \frac{br^4\left(3b(44b^2+3) + 4g[28b^2 + 3 - (4b^2(48b^2+13)+3)\mathcal{R}_0^{-1}]\right)}{30(4b^2+1)} + O(r^6), \tag{11}$$

$$N = 1 + \frac{2r^2}{3}g\,(1 - \mathcal{R}_0) - \frac{16gb^2r^4}{15(4b^2+1)}\left(g(\mathcal{R}_0 - 1)^2 + 3b\mathcal{R}_0\right) + O(r^6), \tag{12}$$

where $\mathcal{R}_0$ is a limiting value of the square root at the origin

$$\mathcal{R} \to \mathcal{R}_0 = \sqrt{1 + 12b^2}. \tag{13}$$

This local solution has a unique free parameter b. In the limiting case $g = 0$ it coincides with that found in the flat space case [17], while to make contact with the corresponding expansion of the EYM theory one has to take the limit $g \to \infty$ with a simultaneous rescaling $b \to b/g$. Near the origin the spacetime is flat.

Like in the EYM case, matching of the local solution departing from the origin as (11,12) and meeting the conditions of asymptotic flatness can be achieved for a discrete sequence of the free parameter values $b = b_n(g)$ labeled by the number n of zeroes of $w(r)$. The proof of existence may be given along the lines of [19, 17], here we do not enter into mathematical details concentrating rather on the qualitative physical picture.

We have investigated numerical solutions in a large range of g. Typical behavior for small g is shown of Figs. (1,2). The YM curves for $g = .001$ are practically undistinguishable from those found previously by Kerner and one of the authors [17] in the flat space BIYM model. The region of oscillations corresponds to an unscreened Coulomb

charge inside the particles. One can see that this region expands both in the direction of small and large r with growing n. The metric deviate from the flat one rather weakly for small g (weak gravity). One observes a stabilization of the metric in the region of the w-oscillations.

The amplitude of oscillations of w in the intermediate region decreases, so in this region the solution can be regarded as approaching the embedded Abelian solution $w \equiv 0$. As was observed by Wirschins, Sood and Kunz [18], the metric for small g also approaches the corresponding Abelian Bion metric [20] (with zero 'seed' mass in the singularity). To avoid confusion it is worth noting that for odd n the Yang-Mills topology of the EBIYM regular solutions (kink) is essentially different form that of the embedded Abelian one $w \equiv 0$ (trivial). Thus it would be misleading to say that the sequence of non-Abelian EBIYM solutions converges to an Abelian one in the global sense.

For large g the regular EBIYM solutions after the coordinate rescaling $r \to \sqrt{g}r$ tend to the BK solutions of the EYM system, see Fig. 3 for $g = 10$. Comparing this with the weak gravity behavior (Fig. 1) we observe that gravity reduces the region of the unscreened charge. The metric deviates substantially form the flat one. The function $N(r)$ for large n exhibits a deep well (an 'almost' horizon), but remains always strictly positive.

The sequences b_n of discrete values of the parameter in the expansion (11) behaves very differently for small and large g. As was found in [17], b_n in the flat space are quite big with respect to the corresponding BK values, and the sequence does not converge with growing n. Contrary to this, the BK sequence b_n rapidly converges to a limiting value b_∞. In this case there is an additional limiting solution with different space-time structure [19].

We have obtained numerically the interpolating functions $b_n(g)$ for several n clearly demonstrating that these two extrema are continuously related indeed (Fig. 5). For small values of g these functions tend to the GK values [17]

$$b_n(0) = b_n^{GK}, \tag{14}$$

while in the opposite limit $g \to \infty$ one recovers the convergent sequence of rescaled BK values [1, 2]

$$b_n^{BK} = \lim_{g\to\infty} b_n(g)/g. \tag{15}$$

The corresponding solutions of the EYM system with the standard YM lagrangian are recovered in terms of the rescaled coordinate $\tilde{r} = r/\sqrt{g}$. In the intermediate region all parameter functions $b_n(g)$ were found to be monotonously varying between the above extrema.

Masses of the regular solutions as functions of the effective gravitational coupling are shown on Fig. 6. For vanishing g one recovers the GK masses after a rescaling

$$\lim_{g\to 0} M_n(g) \to M_n^{GK} g. \tag{16}$$

We recall [17] that the sequence M_n^{GK} converges to the mass of an embedded Abelian solution. The reason is that the main contribution to the mass comes from the region of

oscillations where w approaches an Abelian value $w \equiv 0$ with growing n. But for any n the limiting values $w(0), w(\infty)$ are equal to ± 1, so, as we have already noted, the global topology of non-Abelian solutions is entirely different. In the limit of strong gravity one recovers the BK masses after a rescaling

$$\lim_{g \to \infty} M_n(g) \to M_n^{BK} \sqrt{g}. \tag{17}$$

Now discuss the black holes. These are parameterized by the horizon radius r_h and the value of the YM function $w_h = w(r_h)$ at the horizon. The series expansions near the regular horizon reads:

$$\begin{aligned} w &= w_h + \frac{w_h(w_h^2 - 1)}{r_h N_h'}(r - r_h) + O\left((r - r_h)^2\right) \\ N &= N_h'(r - r_h) + O\left((r - r_h)^2\right), \\ N_h' &= \frac{1}{r_h}\left[1 + 2g^2 r_h^2 \left(1 + \sqrt{1 + (w_h^2 - 1)^2 / r_h^4}\right)\right]. \end{aligned} \tag{18}$$

Asymptotically flat solutions with such boundary condition are likely to exist for any horizon radii r_h. Exterior black hole solutions are very similar to regular solutions, especially for small r_h. So our main interest is in the interior solutions. We start by listing various series solutions that can be obtained near the singularity.

Let us first explore the series expansion for an embedded Abelian solution [20, 21]. For the unit magnetic charge (what corresponds to $w \equiv 0$) the local mass function satisfies the equation

$$m' = g\left(\sqrt{r^4 + 1} - r^2\right). \tag{19}$$

Expanding the square root at small r and integrating one obtains:

$$N = -\frac{2m_0}{r} + 1 - 2g + \frac{2}{3}gr^2 - \frac{1}{5}gr^4 + O(r^8). \tag{20}$$

The 'seed' mass parameter m_0 may be positive, negative or zero. Positive m_0 corresponds to a timelike singularity of the Schwarzschild type. Negative m_0 corresponds to a spacelike singularity, in which case an internal horizon also exist (though contrary to a more common example of the Reissner-Nordström metric, the 'local charge' term Q^2/r^2 now is absent). Vanishing m_0 is particularly interesting. The metric near the origin is then locally flat unless $g = 1/2$ in which case the limiting value $N(0)$ shrinks to zero. For other values of g there is a conical singularity [20], the critical value $g = 1/2$ corresponding to an extremal deficit angle. Within the present framework the embedded Abelian black holes correspond to an identically vanishing function w. It can be shown that the local series solution starting at $r = 0$ with zero initial value $w_0 = 0$ and arbitrary m_0 generates this global embedded Abelian solution.

For non-Abelian solutions the value of the YM function at the singularity $w(0) = w_0$ should therefore be non-zero. We have found the generalization of the series expansion

(20) with $w_0 \neq 0$. It is valid for the non-zero seed mass parameter m_0:

$$w = w_0 \left(1 + \frac{er}{2m_0} + \frac{er^2}{16m_0^2}\left[3(2e-1) - 4ge\right]\right) + O(r^3), \tag{21}$$

$$eN = -\frac{2m_0}{r} + 1 - 2ge + \frac{3gew_0^2 r}{2m_0} + gr^2\left(\frac{2}{3} + \frac{ew_0^2}{12m_0^2}\left[3(5e-2) - 8ge\right]\right) + O(r^3),$$

where $e = 1 - w_0^2$. This local solution has two free parameters w_0, m_0. For $w_0 = 0$ it coincides with (20). If $w_0 \neq 0$ becomes singular in the limit $m_0 = 0$. The search for local solutions with $m_0 = 0$ shows that in this case either $w_0 = 0$ in which case we come back to the Abelian embedded solution $w \equiv 0$, or $w_0 = \pm 1$, then we recover the regular solution (11,12). It is also worth noting that, contrary to the EYM case, in the EBIYM theory there are no local solutions with $N \sim r^{-2}$ behavior at the singularity (Reissner-Nordström type).

Since the asymptotic solution (21) fails to contain a sufficient number of free parameters to be a generic solution, (this number is equal to three for the system of equations (6,7)), the question is how a generic solution looks like near the singularity, in particular, whether it admits any series expansion. In the case of the ordinary Yang-Mills lagrangian such an expandable solution does not exist at all, one finds that a generic solution has a non-analytic oscillating behavior [7]. Here the situation is different, although the generic local solution around the singularity still exhibits non-analyticity in terms of the variable r. It turns out to be series expandable but in terms of the $\sqrt{r}$:

$$\begin{aligned} w &= w_0 + a\sqrt{r} - \frac{a^2 w_0 r}{e} + \frac{a\left(24g^2a^2 - 32g^2a^2e - 3c^2\right)r^{3/2}}{32g^2e^2} - \\ &\quad \frac{a^2\left(\left(16g^2a^2 - 32g^2a^2e - 15c^2\right)w_0 - 8g^2cae\right)r^2}{32g^2e^3} + O(r^{5/2}), \\ N &= 1 - 2\frac{e^2}{a^2 r} + c\sqrt{r} - \frac{a\left(3w_0c + ag^2e\right)r}{e} + O(r^{3/2}). \end{aligned} \tag{22}$$

This local solution contains three free parameters w_0, a, c. The singularity is of the Schwarzschild type, the seed mass m_0 is strictly positive. Two leading terms in expansion of 'kinetic' (negative for $N < 0$) and 'potential' (positive) terms in $\mathcal{R}$ cancel, so the leading behavior in singularity is

$$\mathcal{R} \sim \frac{3c}{4gr^{3/2}}. \tag{23}$$

We have tested for various r_h, g, n that a continuation of the exterior black hole solutions under the horizon meets this generic asymptotic solution indeed. In all numerical experiments the function N remained negative under the horizon and no internal Cauchy horizons were met. Typical global black hole solutions are shown on the Figs. 7,8,9 for $r_h = 1, g = 1, n = 1, 2$. The YM function w outside the horizon has qualitatively the same behavior as in the regular case. Inside the horizon it remains perfectly smooth and tends to a finite limit $w_0 \neq 0$ at the singularity. Recall that for the ordinary quadratic

YM lagrangian the function w inside the horizon of the EYM black holes has a rather sophysticated behavior: while w itself tends to a finite limit w_0 as well, its derivative exhibits a sequence of infinitely increasing absolute values at tiny intervals, whose length tends to zero [7, 8]. Such a behavior causes oscillations of the local mass $m(r)$ with an infinitely increasing amplitude. At the beginning of each oscillation cycle the metric function $N(r)$ takes values very close to zero ('almost' Cauchy horizons), then an exponential growth of $m(r)$ starts. After a few oscillation cycles the maximal values of m attained in subsequent cycles become of the googolplexus order, obviously lying beyond any classical bounds. Contrary to this, in the EBIYM case we observe a smooth $m(r)$ inside the horizon up to the singularity where $m(r)$ has a finite positive value (Fig. 8). Similarly, the second metric function σ tends smoothly to a finite value at the singularity (Fig. 9)

We conclude with the following remarks. The BK solutions were found for the gravity coupled Yang-Mills theory with the usual quadratic lagrangian. Now they were shown to be a strong gravity limit of the gravity coupled BIYM theory which can be interpreted as an effective YM theory including string quantum corrections. Within this theory there exists a limit of decoupled gravity, in which qualitatively similar solutions continue to exist. This shows the way how the BK solutions can be incorporated into the (quantum) string theory.

A particularly interesting implication of this reasoning is the resolution of the problem of 'violent' oscillating singularities typical for hairy EYM black holes. Born-Infeld corrections perfectly regularize the behavior of the metric near the singularity. It is worth noting that the generic singularity is timelike, in conformity with the strong cosmic censorship hypothesis. In numerical experiments we did not observe internal horizons. In principle, such horizons could emerge in pairs, this question is worth to be investigated in more detail.

This work was supported in part by the RFBR grant 00-02-16306.

References

[1] R. Bartnik and J. McKinnon, Phys. Rev. Lett. 61 (1988) 141.

[2] M.S. Volkov and D.V. Gal'tsov, Phys. Reports C 319 (1999) 1, hep-th/9810070.

[3] D.V. Gal'tsov and M.S. Volkov, Phys. Lett. B 273 (1991) 255.

[4] M. S. Volkov, and D. V. Galt'sov, JETP Lett. 50 (1989) 346; Sov. J. Nucl. Phys. 51 (1990) 747.

[5] P. Bizon, Phys. Rev. Lett. 64 (1990) 2844;

[6] H. P. Kunzle, and A. K. M. Masood-ul-Alam, J. Math. Phys. 31 (1990) 928.

[7] E.E. Donets, D.V. Gal'tsov, and M.Yu. Zotov Phys. Rev. D 56 (1997) 3459.

[8] D.V. Gal'tsov, E.E. Donets, and M.Yu. Zotov JETP Lett. 65 (1997) 895.

[9] P. Breitenlohner, G. Lavrelashvili, and D. Maison, Nucl. Phys. B 524 (1998) 427.

[10] J. Polchinski, String Theory, Vols. I and II, CUP, 1998.

[11] A.Tseytlin, Nucl. Phys. B 501 (1997) 41.

[12] J.P. Gauntlett, J. Gomis and P.K. Townsend, JHEP 01 (1998) 003 .

[13] D. Brecher and M.J. Perry, Nucl. Phys. B 527 (1998) 121.

[14] N. Grandi, E.F. Moreno and F.A. Shaposhnik, hep-th/9901073.

[15] J.H. Park, Phys. Lett. B458 (1999) 471.

[16] N. Grandi, R.L. Pakman, F.A. Schaposnik, and G. Silva, Phys. Rev. D 60 (1999) 125002.

[17] D. Gal'tsov, and R. Kerner, Phys. Rev. Lett. (to appear); hep-th/9910171.

[18] M. Wirschins, A. Sood, and J. Kunz, hep-th/0004130.

[19] P. Breitenlohner, P. Forgacs, and D. Maison, Comm. Math. Phys. 163 (1994) 141.

[20] G.W. Gibbons and D.A. Rasheed, Nucl. Phys. B 454 (1995) 185.

[21] D.A. Rasheed, hep-th/9702087.

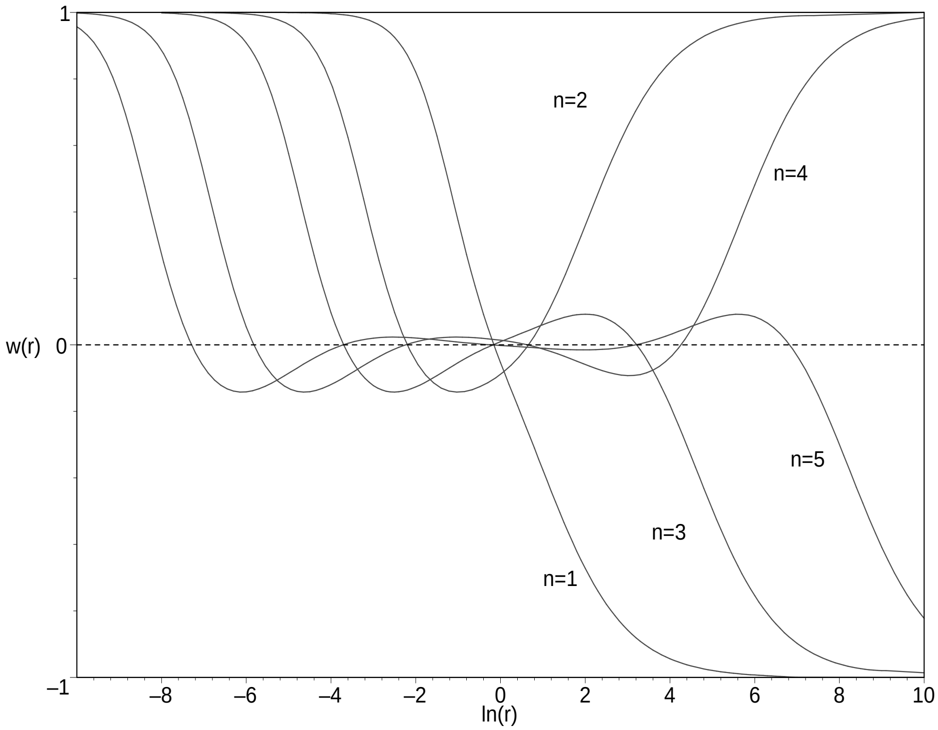

Figure 1: First five solutions $w_n(r)$ for weak gravity ($g = .001$). The oscillation region, which corresponds to the localization of an unscreened magnetic charge inside the BIYM particle, expands with growing n. The amplitude of the first and the last oscillation cycles remains roughly the same, while the amplitude of the intermediate cycles tends to zero with increasing n. All these curves practically coincide with those found in the flat space BIYM theory by Gal'tsov and Kerner.

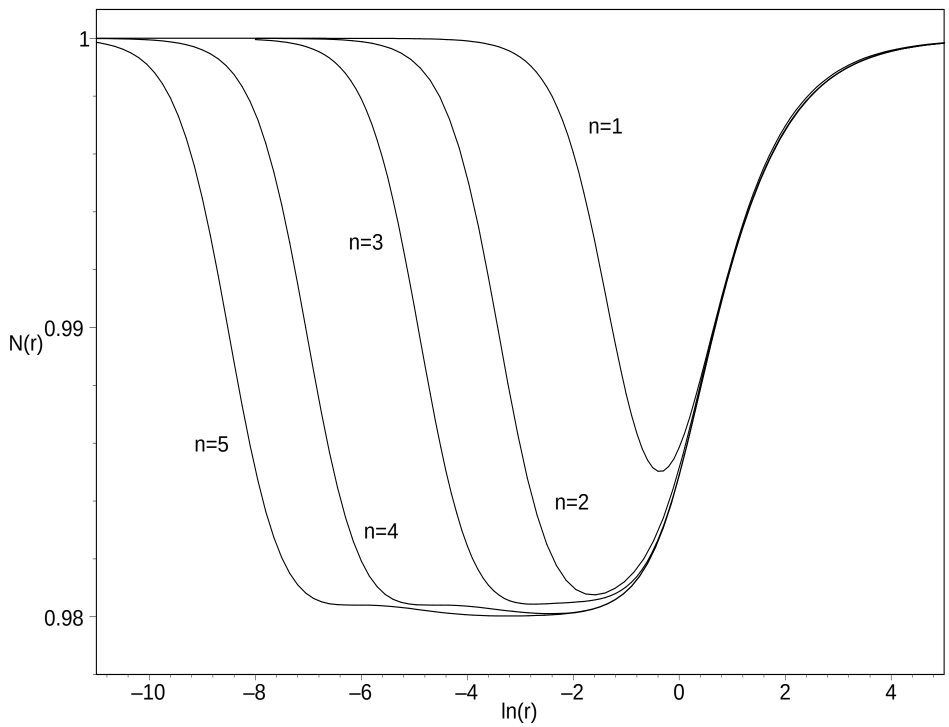

Figure 2: Metric function $N(r)$ for regular solutions with $g = .001$. For all solutions the deviation of the metric from a flat one is within 2%. For higher node numbers n one observes a stabilization of N near the minimal value, this corresponds to the oscillation region of $w(r)$.

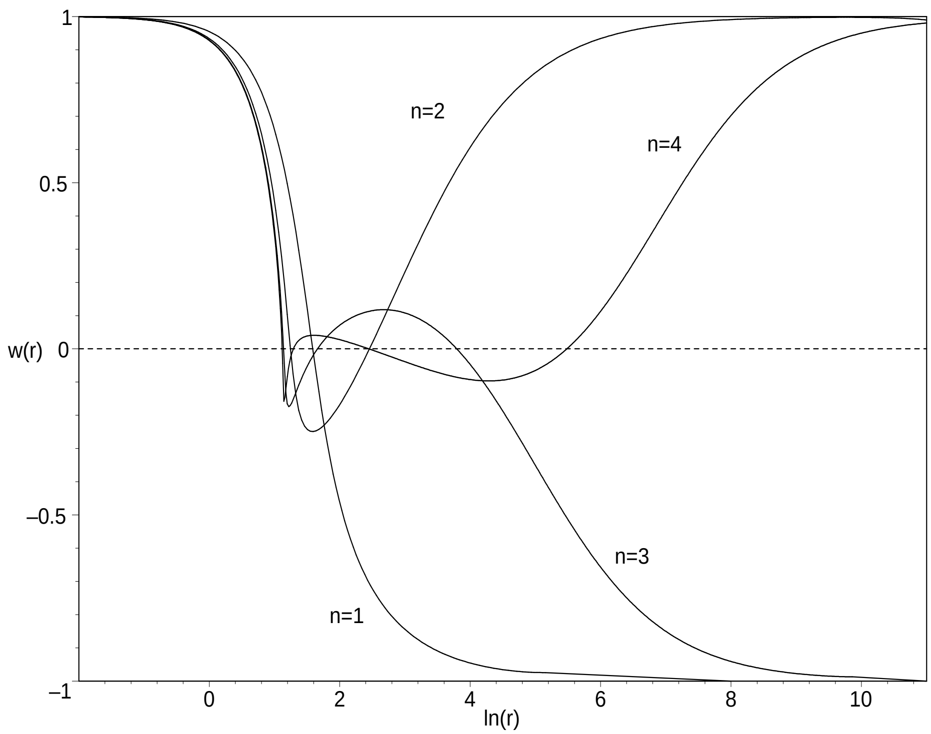

Figure 3: Regular solutions $w_n(r)$ for strong gravity, $g = 10$. These curves are close to BK solutions in terms of a rescaled coordinate $\tilde{r} = r/g^{1/2}$.

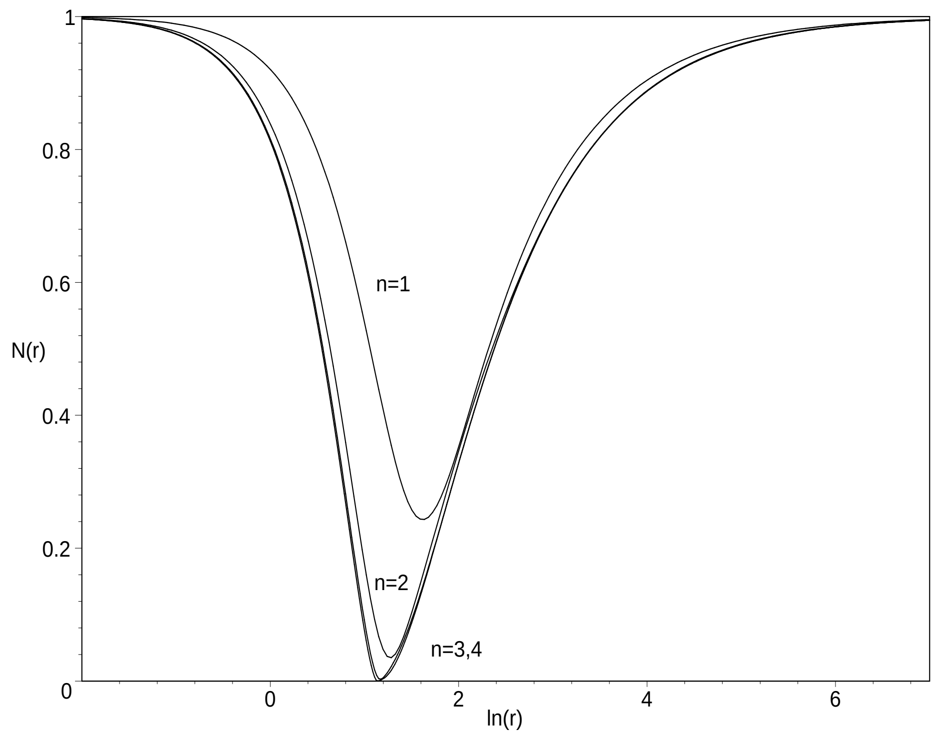

Figure 4: Metric functions $N_n(r)$ for regular solutions with $g = 10$. For higher n the curves come very close to but never reach zero: $N_3^{min} \approx .003$, $N_4^{min} \approx .0003$.

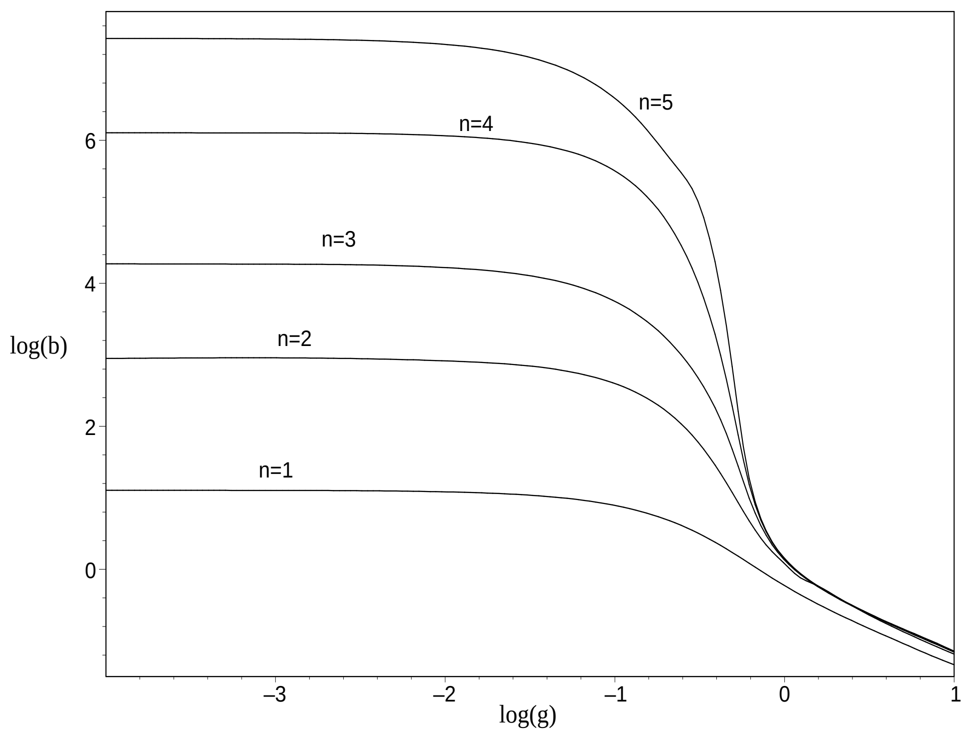

Figure 5: Dependence of the parameters b_n for regular solutions on the effective gravity coupling constant g (in logarithmic coordinates). The left side of the picture correspond to the GK limit. With growing g higher-n curves merge, this is reminiscent of the rapid convergence of the sequence $b_n^{BK} = \lim_{g\to\infty} b_n(g)/g$.

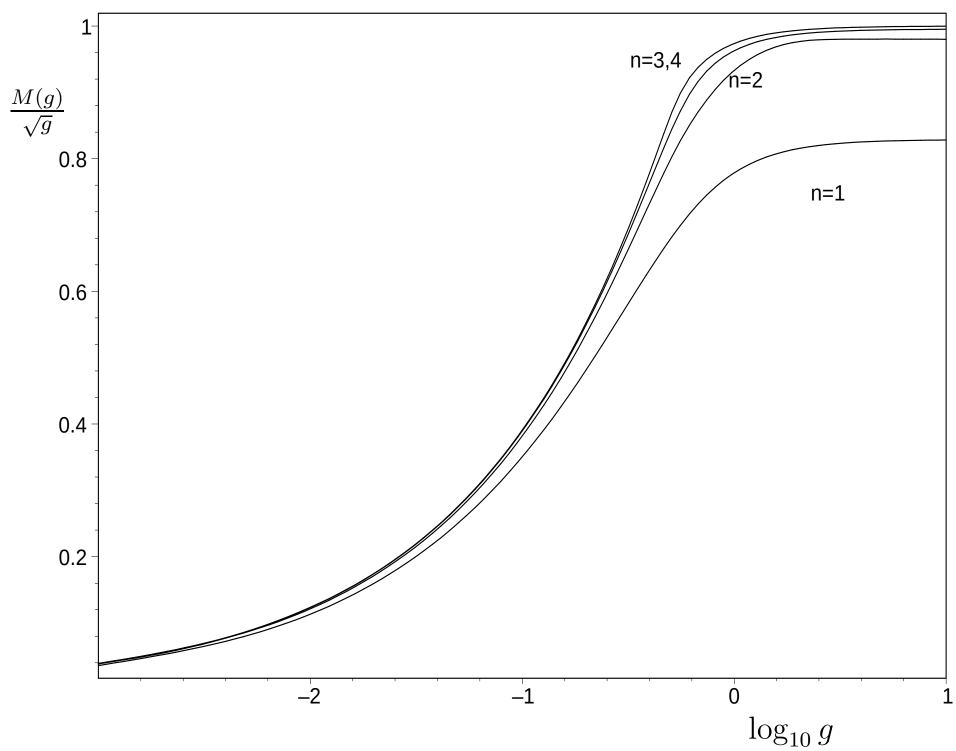

Figure 6: Dependence of masses of regular solutions on the gravity coupling g

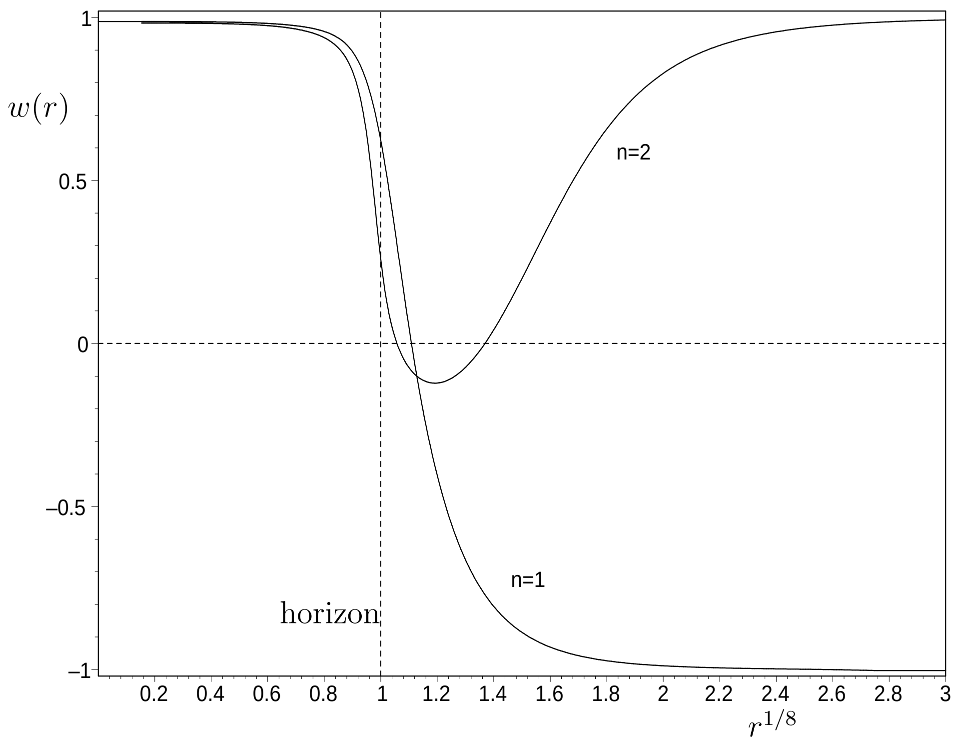

Figure 7: YM function $w(r)$ for $n = 1, 2$ EBIYM black holes with $r_h = 1.0,\ g = 1.0,$. At the singularity these curves tend to constant values non-equal to unity.

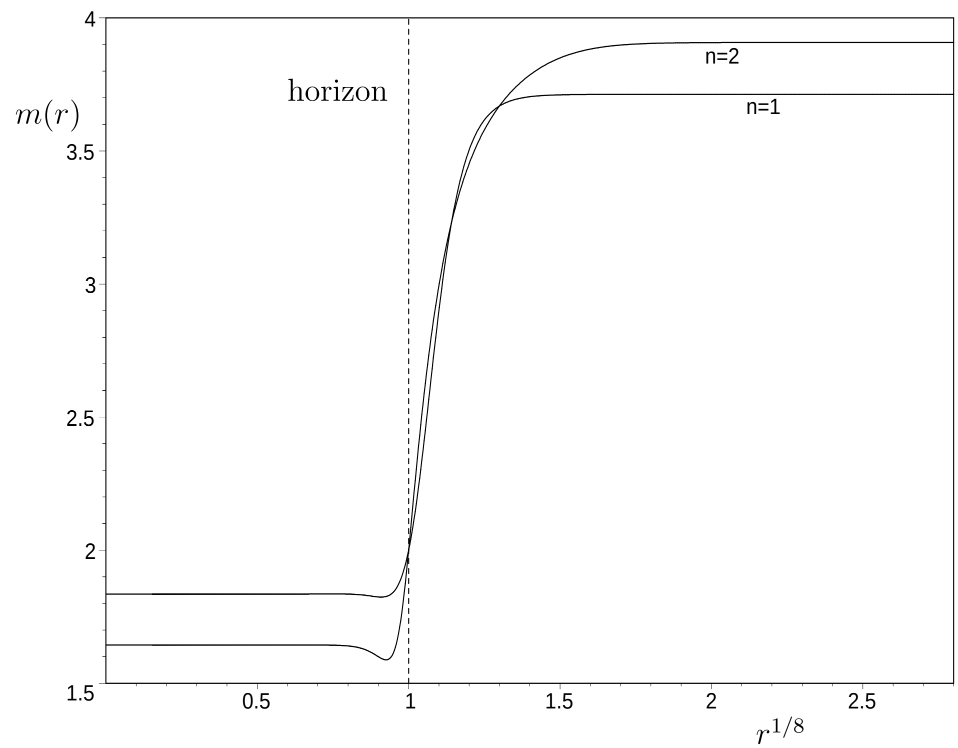

Figure 8: The mass function $m(r)$ function for the EBIYM black holes with $r_h = 1.0,\ g = 1.0,\ n = 1, 2$. Inside the horizon $m(r)$ remains a smooth function tending to a constant limit at the singularity. There are no internal horizons, the singularity is timelike.

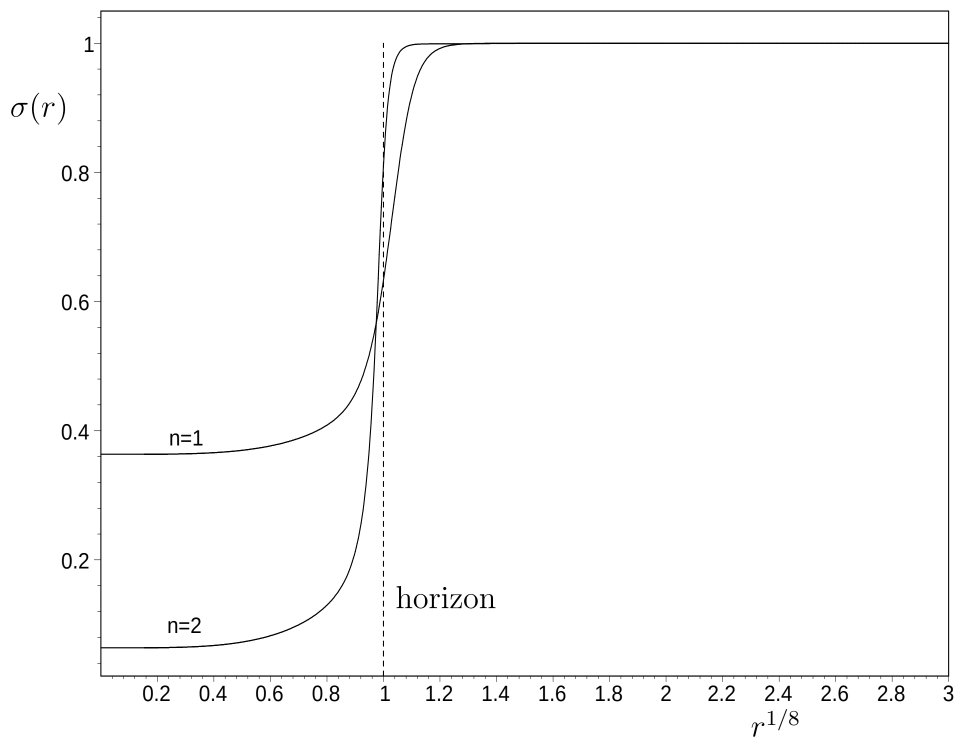

Figure 9: The metric function $\sigma(r)$ for the EBIYM black holes with $r_h = 1.0,\ g = 1.0,\ n = 1, 2$.